Review for Final Exam

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Seminar in Engineering Analysis

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California State University Northridge

Review for Final

- Monday, December 11, 8-10 pm
- Open book and notes, similar to two midterm exams
- Will be cumulative, but will have greater weight on material since second midterm
  – Matrix and eigenvalue problems (including simultaneous linear equations)
  – Ordinary differential equations including Laplace Transforms
  – Numerical solutions of ODEs

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Review Matrix Basics

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} \end{bmatrix}$$

- Array of numbers with n rows and m columns
- Components are $a_{(row)(column)}$
- Size of matrix (n x m) is number of rows and columns
- Square matrix: m = n

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Review Multiplying Matrices

- For matrix multiplication, **C** = **AB**
  – **A** has n rows and p columns
  – **B** has p rows and m columns
  – **C** has n rows and m columns

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad (i = 1, n;\ j = 1, m)$$

- For **C** = **AB**, we get c_{ij} by adding products of terms in row i of **A** (left matrix) by terms in column j of **B** (right matrix)
- $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} + \ldots$
- In general, **AB** ≠ **BA**
- “Premultiply” by matrix on left and “post-multiply” by matrix on right

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Review General Determinants

- Any size determinant can be evaluated by any of the following equations

$$Det\,\mathbf{A}_{(n\,x\,n)} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Minors, M — Cofactors C

- Can pick any row or any column
- Choose row or column with several zeros
- Can apply equation recursively; evaluate a 5 x 5 determinant as a sum of 4 x 4 determinants then get 4 x 4’s in terms of 3 x 3’s

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Review Inverse of a Matrix

- For square matrix, **A**, the inverse, $\mathbf{A}^{-1}$, if it exists, gives $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- Find the components of $\mathbf{B} = \mathbf{A}^{-1}$, b_{ij}, from determinant of **A** and its cofactors

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad b_{ij} = \frac{A_{ji}}{Det(\mathbf{A})} = (-1)^{i+j} \frac{M_{ji}}{Det(\mathbf{A})}$$

- Use this formula to get algebraic equations for components of inverse matrix not for numerical analysis

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Review Norms

***q* norm definition:** $\|\mathbf{x}\|_q = \left[\sum |x_i|^q\right]^{1/q}$

- Norm of vector **x** expressed as ||x|| generalizes notion of vector length
- q norm is one possible norm definition
  - usual vector length is the "two norm", $\|\mathbf{x}\|_2$
  - one norm is sum of absolute values
  - infinity norm is the element with maximum absolute value

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Review Inner Products

- General expression is (**x, y)**
- For two conventional vectors, [x_1 x_2 x_3 … x_n] and [y_1 y_2 y_3 … y_n], the inner product is $\Sigma x_i y_i$
- For two column vectors, **x** and **y**, we can express the inner product as $\mathbf{x}^T\mathbf{y}$
- For two row vectors, **x** and **y**, we can express the inner product as $\mathbf{x}\mathbf{y}^T$
- We can also define inner products as integrals of two functions

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Review Linear (In)dependence

- A set of vectors ***linearly dependent*** if the following equation holds, where at least one of the α_i is not equal to zero.

$$\alpha_1 \mathbf{x}_{(1)} + \alpha_2 \mathbf{x}_{(2)} + \cdots + \alpha_k \mathbf{x}_{(k)} = \sum_{i=1}^{k} \alpha_i \mathbf{x}_{(i)} = \mathbf{0}$$

- A linearly independent set of vectors is one that is not linearly dependent.
- Cannot have $\mathbf{x}_{(i)} = 0$ in LI set

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General Gauss Elimination

- Use each row from row 1 to row n-1 as the "pivot" row
  - Work on each row below the pivot row
    - Multiply pivot row by $a_{row,pivot}/a_{pivot,pivot}$
    - Subtract result from row r (r = pivot+1 to n) to create modified rows where $a_{row,pivot} = 0$
    - Operation requires subtraction for each column of **A** right of pivot column and for **b**
  - Repeat for each row below pivot
- Repeat for rows 1 to n-1 as pivot rows

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Rank and Linear Equations

- Gauss elimination for solving equations and determining rank (number of linearly independent rows or columns)
- Solution of **Ax** = **b**
  - No solutions unless rank **A** = rank [**A b**]
  - Unique if rank **A** = rank [**A b**] = number of unknowns (infinite if rank < unknowns)
  - Homogenous equations, **Ax** = **0**: only solution is **x** = **0** unless Det **A** = 0 (same as saying Rank **A** < n)
  - Find rank with Gaussian elimination

Rank = number of nonzero rows

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Eigenvalues/Eigenvectors

- Basic definition (**A** n x n)**: Ax** = λ**x**
- Det [**A** – **I**λ] = 0 gives n^{th} order equation for eigenvalues
  - n eigenvalues (may not be distinct)
  - solve [**A** – **I**λ]**x** = **0** for n components of each of n eigenvectors
  - eigenvectors undetermined to within a multiplicative constant
  - eigenvectors may or may not be linearly independent

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Condition of a Matrix

- In solution of **Ax** = **b,** ||**r**|| = ||**b** – **Ax** || is numerical residual we can calculate

$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \le \|\mathbf{A}^{-1}\|\|\mathbf{A}\|\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} = \kappa(\mathbf{A})\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

- Condition number $\kappa(\mathbf{A}) \equiv \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|$
- Small is < 10; large is about 100 or more
- Expect large condition numbers to create problems in numerical solutions
  - Use pivoting to reduce numerical error

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ODE Classifications

- Third-order, linear, homogenous $\frac{d^3y}{dx^3} + \sin(x)\frac{dy}{dx} - x^2 y = 0$
- Second-order, non-linear, homogenous $\frac{d^2y}{dx^2} + \sin(y) = 0$
- Second-order, linear, non-homogenous $\frac{d^2y}{dx^2} + y = e^x \cos(x)$
- Third-order, non-linear, non-homogenous $\frac{d^3y}{dx^3} + y\frac{dy}{dx} = 1$

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Separable ODE Forms

- Simple differential equations can be written as integrals
  - Even if numerical quadrature is required this is more accurate than numerical solution of ODE

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad y = \int f(x)dx + C$$

$$\frac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \int g(y)dy = \int f(x)dx + C$$

$$\frac{dy}{dx} = h\left(\frac{y}{x}\right) \quad \Rightarrow \quad \int \frac{dx}{x} = \int \frac{du}{h(u)-u} + C$$

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First-order ODEs

- First order rate equation where rate is proportional to amount dy/dt = -ky
- $y = y_0 e^{-k(t-t_0)}$
- General linear first order equation for y(x): dy/dx + f(x)y = g(x) has closed form solution shown below
- C is found from initial condition

$$p = \int f(x)dx \qquad y = e^{-p}\left[C + \int e^p g(x)dx\right]$$

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Review y'' + αy' + βy = 0

- Three cases depending on $\omega^2 = \beta - \alpha^2/4$
- Double root when $\beta = \alpha^2/4$:
  - $y = (C_1 + C_2 x)\, e^{-\alpha x/2}$
- Complex roots when $\beta > \alpha^2/4$, $\omega^2 > 0$
  - $y = e^{-\alpha x/2}\,[A\cos \omega x + B\sin \omega x]$
- Distinct real roots when $\beta < \alpha^2/4$
  - $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}$$

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Review y'' + αy' + βy = 0 II

- Initial conditions $y(0) = y_0$ and $y'(0) = v_0$
- Double root when $\beta = \alpha^2/4$:
  - $y = [(v_0 + y_0\alpha/2)x + y_0]\, e^{-\alpha x/2}$
- Complex roots when $\beta > \alpha^2/4$
  - $y = e^{-\alpha x/2}\,[y_0\cos \omega x + \omega^{-1}(v_0 + y_0\alpha/2)\sin \omega x]$
- Distinct real roots when $\beta < \alpha^2/4$
  - $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
  - $C_1 = (\lambda_2 y_0 - v_0)/(\lambda_2 - \lambda_1)$
  - $C_2 = (v_0 - \lambda_1 y_0)/(\lambda_2 - \lambda_1)$

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Review Nonhomogeneous ODEs

- Homogenous: $f(d^ny/dx^n,\ldots y) = 0$
- Nonhomogeneous: $f(d^ny/dx^n,\ldots y) = r(x)$
- First solve homogenous part: y_H
  - Do not find constants in this solution
- Find particular solution, y_P, using the method of undetermined coefficients
- Combine parts to get $y = y_H + y_P$
- Apply initial/boundary conditions to y to find undetermined constants

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Review Undetermined Coefficients

- Used for constant coefficient equation $y'' + ay' + by = r(x)$ (or higher order)
- Solution is $y = y_P + y_H$, where y_H is solution of $y_H'' + ay_H' + by = 0$
- Postulate a solution for y_P following guidelines on next two charts
- Plug solution into ODE and solve for unknown coefficients
  - Overall coefficients of like terms on both sides of ODE must vanish

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Table of Trial y_P Solutions

For these r(x)	Start with this y_P
$r(x) = Ae^{ax}$	$y_P = Be^{ax}$
$r(x) = Ax^n$	$y_P = a_0 + a_1x + \ldots + a_nx^n$
$r(x) = A\sin\omega t$	$y_P = B\sin\omega t + C\cos\omega t$
$r(x) = A\cos\omega t$	
$r(x) = Ae^{ax}\sin\omega t$	$y_P = e^{ax}(B\sin\omega t + C\cos\omega t)$
$r(x) = Ae^{ax}\cos\omega t$	

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Special Rules

- If the right-hand-side, r(x) consists of more than one term from the previous table, use a y_P that contains all the corresponding y_P terms
  - For $r(x) = A\cos bx + Ce^{dx}$, use $y_P = E\sin bx + F\cos bx + Ge^{dx}$
- If r(x) is proportional to a solution for the homogenous equation, use y_P equal to x times the y_P shown in the table
  - For a double root, multiply table y_P by x^2

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Review Higher Order ODEs

- n^{th} order ODE with constant coefficients
- Solution is $y = y_H + y_P$

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = r(x)$$

- Homogenous solution $y_H = \sum_{k=1}^{n} C_k e^{\lambda_k x}$
- λ_k are solutions to the equation $\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0$
- Multiple and complex roots

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Review Higher Order ODEs II

- y_P may be found by undetermined coefficients or variation of parameters
  - Use same process for method of undetermined coefficients
  - May have e^{ax}, sin ax, or cosine ax in r(x) where a root of homogenous solution
    - Must handle possibility of multiple roots in higher order equations
  - Variation of parameters is more complex and will not be on final exam

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Matrix Differential Equations

- Matrix components in $\frac{d\mathbf{y}}{dt} + \mathbf{Ay} = \mathbf{r}$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} \quad \frac{d\mathbf{y}}{dt} = \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} dy_1/dt \\ dy_2/dt \\ dy_3/dt \\ \vdots \\ \vdots \\ dy_n/dt \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{r} = [r_1 \quad r_2 \quad r_3 \quad \cdots \quad \cdots \quad r_n]^T$$

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Solving $\frac{d\mathbf{y}}{dt} + \mathbf{Ay} = \mathbf{r}$

- Possible conversion when $\mathbf{A}_{(n \times n)}$ has n linearly independent eigenvectors which are the columns of matrix, **X**
- Define a new vector **s** = $\mathbf{X}^{-1}\mathbf{y}$ (**y** = **Xs**)

$$\frac{d\mathbf{Xs}}{dt} + \mathbf{AXs} = \mathbf{r} \qquad \mathbf{X}^{-1}\frac{d\mathbf{Xs}}{dt} + \mathbf{X}^{-1}\mathbf{AXs} = \mathbf{X}^{-1}\mathbf{r}$$

$$\frac{d\mathbf{s}}{dt} + \mathbf{X}^{-1}\mathbf{AXs} = \mathbf{p} \qquad \frac{d\mathbf{s}}{dt} + \mathbf{\Lambda s} = \mathbf{p}$$

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Matrix Solution Terms $s_i = C_i e^{-\lambda_i t} + q_i$

$$\mathbf{E}(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & e^{-\lambda_3 t} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & e^{-\lambda_n t} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_n \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ \vdots \\ q_n \end{bmatrix}$$

- With these definitions, $s_i = C_i e^{\lambda_i t} + q_i$ becomes **s** = **E**(t)**C** + **q** (at t = 0, **E**(0) = **I**)

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Solution for vector y

- **y** = **Xs** = **XEC** + **Xq**
- Setting **y** = $\mathbf{y}_0$ at t = 0 we gives **C** = $\mathbf{X}^{-1}\mathbf{y}_0 - \mathbf{q}_0$ so that we find the solution to the matrix system as
- Result: **y** = **XE** [$\mathbf{X}^{-1}\mathbf{y}_0 - \mathbf{q}_0$] + **Xq**
- For homogenous systems (**q** = **0**), the solution is **y** = $\mathbf{XEX}^{-1}\mathbf{y}_0$

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Laplace Transform Definition

- Transforms from a function of time, f(t), to a function in a complex space, F(s), where s is a complex variable
- The transform of a function, is written as F(s) = $\mathcal{L}$f(t) where $\mathcal{L}$ denotes the Laplace transform (use $\Im$ for $\mathcal{L}$ in some equations)
- Laplace transform defined as the following integral

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$$

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Simple Laplace Transforms

f(t)	F(s)	f(t)	F(s)
t^n	$n!/s^{n+1}$	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
t^x	$\Gamma(x+1)/s^{x+1}$		
e^{at}	$1/(s - a)$	$e^{at}\cos \omega t$	$\frac{(s-a)}{(s-a)^2+\omega^2}$
sin ωt	$\omega/(s^2 + \omega^2)$		
cos ωt	$s/(s^2 + \omega^2)$	Additional transforms in pp 264-267/248-251 of Kreyszig 9th/10th edition	
sinh ωt	$\omega/(s^2 - \omega^2)$		
cosh ωt	$s/(s^2 - \omega^2)$		

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Review Transforms Properties

- $\mathscr{L}[af_1(t) + bf_2(t)] = a\mathscr{L}[f_1(t)] + b\mathscr{L}[f_2(t)]$
- First shifting theorem
- If $\mathscr{L}[f(t)] = F(s)$ then $\mathscr{L}[e^{at}f(t)] = F(s – a)$
  – Example: $\mathscr{L}[\cos(\omega t)] = s/(s^2 + \omega^2)$ so $\mathscr{L}[e^{at}\cos(\omega t)] = (s – a)/[(s – a)^2 + \omega^2]$
- Derivative transforms where $\mathscr{L}[f(t)] = F(s)$
  – $\mathscr{L}[df/dt] = sF(s) – f(0)$
  – $\mathscr{L}[d^2f/dt^2] = s^2F(s) – sf(0) – f'(0)$
  – Similar results for higher derivatives

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Solving Differential Equations

- Transform all terms in the differential equation to get an algebraic equation
  – For a differential equation in y(t) we get the transforms $Y(s) = \mathscr{L}[y(t)]$
  – Similar notation for other transformed functions in the equation $R(s) = \mathscr{L}[r(t)]$
- Solve the algebraic equation for Y(s)
- Obtain the inverse transform for Y(s) from tables to get y(t)
  – Use method of partial fractions to get from Y(s) equation to transforms in tables

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Review Partial Fractions

- Method to convert fraction with several factors in denominator into sum of individual factors (in denominator)
- Example is F(s) = 1/(s+a)(s+b)
- Write 1/(s+a)(s+b) = A/(s+a) + B/(s+b)
- Multiply by (s+a)(s+b) and equate coefficients of like powers of s
  – 1 = A(s + b) + B(s + a)
  – A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms

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Review Partial Fractions II

  – A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms
  – Solving for A and B gives A = -B = 1/(b – a)
- Result: 1/(s+a)(s+b) = 1/[(b – a)(s + a)] – 1/[(b – a)(s + b)]
  – So $f(t) = [e^{-at} – e^{-bt}]/(b – a)$
- This actually matches a table entry
- Follow same basic process for more complex fractions
- Special rules for repeated factors and complex factors

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Review Partial Fraction Rules

- Repeated fractions for repeated factors

$$\frac{1}{\cdots(s+a)^n\cdots} = \cdots + \frac{A_n}{(s+a)^n} + \frac{A_{n-1}}{(s+a)^{n-1}} + \cdots + \frac{A_2}{(s+a)^2} + \frac{A_1}{s+a} + \cdots$$

- Complex factors $(s + \alpha – i\beta)(s + \alpha + i\beta)$

$$\frac{1}{\cdots(s+\alpha-i\beta)(s+\alpha+i\beta)\cdots} = \cdots + \frac{As+B}{(s+\alpha)^2+\beta^2} + \cdots$$

- Pure imaginary factor

$$\cdots + \frac{1}{s^2+\beta^2} + \cdots = \frac{1}{\cdots(s-i\beta)(s+i\beta)\cdots} = \cdots + \frac{As+B}{s^2+\beta^2} + \cdots$$

- Real squared factors

$$\cdots + \frac{1}{s^2-\beta} + \cdots = \cdots + \frac{A}{\left(s+\sqrt{\beta}\right)} \frac{B}{\left(s-\sqrt{\beta}\right)} + \cdots$$

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Truncation Error

- If we truncate series after m terms

$$f(x) = \underbrace{\sum_{n=0}^{m} \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=a} (x-a)^n}_{\text{Terms used}} + \underbrace{\sum_{n=m+1}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=a} (x-a)^n}_{\text{Truncation error, } \varepsilon_m}$$

Terms used   Truncation error, ε_m

- Truncation error as single term at unknown location

$$\varepsilon_m = \sum_{n=m+1}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=a} (x-a)^n = \frac{1}{(m+1)!} \frac{d^{m+1} f}{dx^{m+1}}\bigg|_{x=\xi} (x-a)^{m+1}$$

- Derive finite-differences for derivatives

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First Derivative Expressions

First order forward $f_i' = \frac{f_{i+1} - f_i}{h} + O(h)$

First order backward $f_i' = \frac{f_i - f_{i-1}}{h} + O(h)$

Second order central $f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$

Second order forward $f_i' = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h} + f_\xi''' \frac{h^2}{3}$

Second order backwards $f_i' = \frac{f_{i-2} - 4f_{i-1} + 3f_i}{2h} + f_\xi''' \frac{h^2}{3}$

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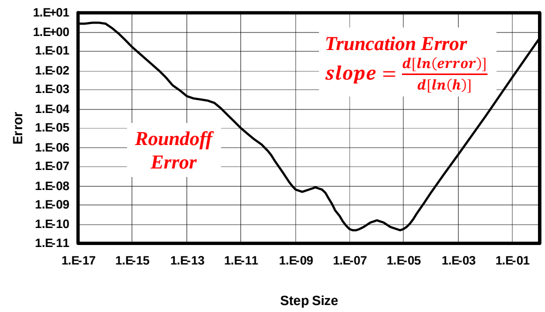
Effect of Step Size on Error



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Simple Numerical ODE $\frac{dy}{dt} = f(t,y)$

- Euler: y_i+1 = $y_i + h_i f_i = y_i + h_i\, f(t_i, y_i)$
- Huen's method

$$y_{i+1}^0 = y_i + h_{i+1} f(t_i, y_i) \qquad t_{i+1} = t_i + h_{i+1}$$

$$y_{i+1} = y_i + \frac{h_{i+1}}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)\right] = \frac{y_i + y_{i+1}^0 + h_{i+1} f(t_{i+1}, y_{i+1}^0)}{2}$$

- Modified Euler method

$$y_{i+\frac{1}{2}} = y_i + \left[\frac{h_{i+1}}{2}\right] f(t_i, y_i) \qquad t_{i+\frac{1}{2}} = t_i + \frac{h_{i+1}}{2}$$

$$y_{i+1} = y_i + h_{i+1} f(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

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Fourth-Order Runge-Kutta

- Uses four derivative evaluations per step

$$y_{i+1} = y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \qquad t_{i+1} = t_i + h_{i+1}$$

$$k_1 = h_{i+1} f(t_i, y_i)$$

$$k_2 = h_{i+1} f\left(t_i + \frac{h_{i+1}}{2}, y_i + \frac{k_1}{2}\right)$$

$$k_3 = h_{i+1} f\left(t_i + \frac{h_{i+1}}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = h_{i+1} f(t_i + h_{i+1}, y_i + k_3)$$

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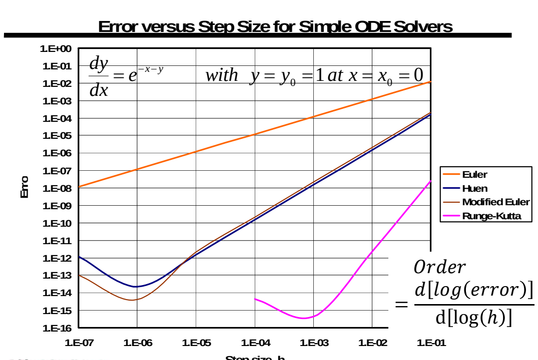

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Adams-Bashforth-Moulton

- Predictor corrector method
- Predictor equation uses four points

$$y_{n+1}^P = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$

- Corrector equation uses four points including point n+1 with predicted y^P

$$y_{n+1}^C = y_n + \frac{h}{24}\left(9 f\left(x_{n+1}, y_{n+1}^P\right) + 19 f_n - 5 f_{n-1} + f_{n-2}\right)$$

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Adams-Bashforth-Moulton II

- Use difference between predictor and corrector to estimate solution error, E_C

$$E_C = \left| \frac{19}{270} \left(y_{n+1}^C - y_{n+1}^P \right) \right|$$

- If error estimate, EC, is in allowed range do not change step size
- Double step size if EC is too large
- Half step size if EC is small enough

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Explicit vs. Implicit Methods

- Explicit algorithms use information from previous steps to get values at new time step, t_{n+1}
  - Algorithms may compute trial values at new time step to get their result
- Implicit algorithms use values at new time step in their basic equation
- Require iterations for each time step or series expansion of derivatives for simple methods

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Numerical ODE Solutions

- Algorithms for initial-value problem (IVP) with single first order equation
- Apply to system of first-order equations
- Convert higher order IVP equations to system of first order equations
- Convert system of higher order IVP ODEs to system of first order equations

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Boundary-value problems

- Shoot-and-try is usually best approach for non-linear problems
  - Adapts methods used for initial-value problems
- Finite-difference approach usually best for linear problems
  - Application to ODEs is instructive of basic use of these methods for PDEs
- Finite-elements important for engineering design software not on final

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Eigenvalue Problems

- Finite difference equations for eigenvalue problems must be solved for matrix eigenvalues
- ODE eigenvalues are parameters that must be fitted because there are not enough arbitrary constants in the ODE solution to fit all the boundary conditions
  - May be unknown parameters in the problem formulation

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Final Exam Dec 11, 8-10 pm

- Open book and notes, including homework solutions
  - Comprehensive but more problems on numerical analysis not covered on midterms
- Make your own notes to use for exam
  - You are in trouble if you have to use the book on an open-book exam
- More credit given for showing how to obtain solution than for providing final details of algebra or arithmetic

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